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Ultra-violet Finite Noncommutative Theories

I. Jack and D.R.T. Jones

Dept. of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK

We establish the ultra-violet finiteness of various classes of noncommutative gauge theories.

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There has been a great deal of recent interest in noncommutative (NC) quantum field theories, stimulated by a connection with string theory and M-theory; see for example Refs. [1]–[19]. The theories have, moreover, novel properties which make them worthy of attention in their own right; for example NC quantum electrodynamics exhibits both asymptotic freedom and charge quantisation.

The algebra of functions on a noncommutative space is isomorphic to the algebra of functions on a commutative space with coordinates x^μ, with the product $f * g(x)$ defined as follows

$$f * g(x) = e^{-i\Theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}} f(x+\xi)g(x+\eta)|_{\xi,\eta\to 0}, \tag{1}$$

where Θ is a real antisymmetric matrix. Quantum field theories analogous to the corresponding commuting theories are now straightforward to define, with $*$-products replacing ordinary products. In the case of gauge theories there are a number of subtleties, however. Consider a field $\phi(x)$ which transforms as follows under a local symmetry transformation:

$$\phi(x) \to \phi'(x) = U(x) * \phi(x) = e_*^{i\Lambda(x)} * \phi(x), \tag{2}$$

where

$$e_*^{i\Lambda(x)} = 1 + i\Lambda - \frac{1}{2!}\Lambda * \Lambda + \cdots \tag{3}$$

By considering the product $U_1 * U_2 = e_*^{i\Lambda_1} * e_*^{i\Lambda_2}$ it is easy to show that SU_N is not a group under the $*$-product, whereas U_N is, so that we will devote our attention to U_N gauge theories. Such gauge theories are constructed using the gauge fields A_μ and matter fields χ, ξ, ϕ (scalars or fermions) transforming as follows:

$$A'_\mu = U * A_\mu * U^{-1} + ig^{-1} U * \partial_\mu U^{-1} \tag{4a}$$

$$\chi' = U * \chi \tag{4b}$$

$$\xi' = \xi * U^{-1} \tag{4c}$$

$$\phi' = U * \phi * U^{-1} \tag{4d}$$

where χ, ξ, ϕ transform according to the fundamental, the anti-fundamental and the adjoint representations respectively. One may also, of course, have matter singlets; but, as has been noted by previous authors, it is not clear how to construct other representations (such as fractionally charged particles in the U_1 case).

In this paper we consider the ultra-violet (UV) divergences of NC theories, and in particular seek theories that are UV finite. Consider the pure (no matter) U_N NC gauge theory (NCGT). If one computes the one loop corrections and isolates the UV divergence, one finds that this can be described both for $N = 1$[2][3][4][6] and for $N \geq 2$ [3][4][13][18] by a single β-function β_g, which is moreover identical (for $N \geq 2$) to the corresponding one-loop β_g for the SU_N commutative theory (CGT). (Contrast this to the U_N CGT case, where of course, writing $U_N \equiv SU_N \otimes U_1$, the U_1 gauge coupling is unrenormalised). Although our chief interest here is in supersymmetric theories, an elementary consequence of our methods is that for the pure U_N gauge theory, the NCGT β_g is *to all orders* identical to the large N approximation to the corresponding SU_N CGT β_g.

The NC formalism extends readily to supersymmetric theories[1]. An $\mathcal{N} = 1$ U_N gauge theory with a set of adjoint chiral superfields Φ_i is described by the Lagrangian

$$L = \int d^4\theta \, \mathrm{Tr}\left(e_*^{-gV} * \overline{\Phi}_i * e_*^{gV} * \Phi_i\right) + \left[\int d^2\theta \left(W(\Phi_i) + \tfrac{1}{4} W^\alpha * W_\alpha\right) + \mathrm{c.c.}\right], \tag{5}$$

where V is the vector superfield, W^α the corresponding field strength, and the superpotential $W(\Phi_i)$ is holomorphic and gauge invariant.

We will focus particularly on the following two theories:

$$W_1 = h_1 \mathrm{Tr}\left(\Phi_1 * [\Phi_2, \Phi_3]_*\right) = h_1(W_a - W_b) \tag{6}$$

$$W_2 = h_2 \mathrm{Tr}\left(\Phi_1 * \{\Phi_2, \Phi_3\}_*\right) = h_2(W_a + W_b) \tag{7}$$

where $W_a = \mathrm{Tr}(\Phi_1 * \Phi_2 * \Phi_3)$ and $W_b = \mathrm{Tr}(\Phi_1 * \Phi_3 * \Phi_2)$, and $\Phi_{1\cdots 3}$ are adjoint chiral supermultiplets. If we define

$$\Phi = \frac{1}{\sqrt{2}} \phi^a \lambda^a, \quad a = 0, 1, \cdots N^2 - 1 \tag{8}$$

where $[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c$, $\{\lambda^a, \lambda^b\} = 2d^{abc}\lambda^c$, and $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$, then in the commutative versions of the above theories we would have

$$W_1^C = i\sqrt{2} h_1 f^{abc} \phi_1^a \phi_2^b \phi_3^c \tag{9}$$

and

$$W_2^C = \sqrt{2} h_2 d^{abc} \phi_1^a \phi_2^b \phi_3^c \tag{10}$$

[1] Note in particular that in Ref. [10] the gauge invariance of the one loop effective action for the $\mathcal{N}$ =4 theory was demonstrated

and it is interesting to contrast this with the NC case where we have

$$W_1 = \frac{h_1}{\sqrt{2}} \left(d^{abc} \phi_1^a * \left[\phi_2^b, \phi_3^c \right]_* + i f^{abc} \phi_1^a * \left\{ \phi_2^b, \phi_3^c \right\}_* \right) \tag{11}$$

and

$$W_2 = \frac{h_2}{\sqrt{2}} \left(d^{abc} \phi_1^a * \left\{ \phi_2^b, \phi_3^c \right\}_* + i f^{abc} \phi_1^a * \left[\phi_2^b, \phi_3^c \right]_* \right) \tag{12}$$

In both the CGT and the NCGT cases, W_1 corresponds to $\mathcal{N} = 4$ supersymmetry, if we set $h_1 = g$. It is well-known that the $\mathcal{N} = 4$ CGT is all orders finite[2]; as we shall see the same is true in the NCGT $\mathcal{N} = 4$ case. This is to be expected since in general NC theories have improved UV divergence properties. Somewhat more surprising, however, is the following: in the CGT case, the SU_N version of W_2^C, for the case

$$h_2 = gN/\sqrt{N^2 - 4} \tag{13}$$

is the so-called $\mathcal{N} = 4d$ model discussed in Refs. [20], [21]. It is UV finite through two loops, but has a three (and higher) loop divergence [22], which can, however, be removed [23] by replacing Eq. (13) by

$$h_2 = gN/\sqrt{N^2 - 4)} + a_5 g^5 + \cdots \tag{14}$$

where $a_5, \cdots$ are calculable constants. In the NCGT case the U_N version of the theory is, as we shall see, all orders UV finite simply given $h_2 = g$, in other words without recourse to the kind of coupling constant redefinition represented by Eq. (14).

Since the Φ are adjoint fields in U_N we can use the diagrammatic notation originally introduced by 't Hooft [24], where we represent Φ_b^a by a double line as in Fig. 1, the arrow pointing towards the upper index. This is in fact a considerable simplification compared to the generalised f^{abc}, d^{abc} formulation that has been used in some papers.

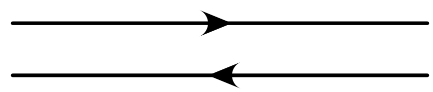

Fig. 1: The propagator for an adjoint $U(N)$ field

[2] The $\mathcal{N} = 4$ U_N CGT consists of the direct product of the familiar $\mathcal{N} = 4$ SU_N theory with a $\mathcal{N} = 4$ U_1 free field theory

The vertices W_a, W_b, and their complex conjugates $\overline{W}_a$ and $\overline{W}_b$ are then represented as in Fig. 2.

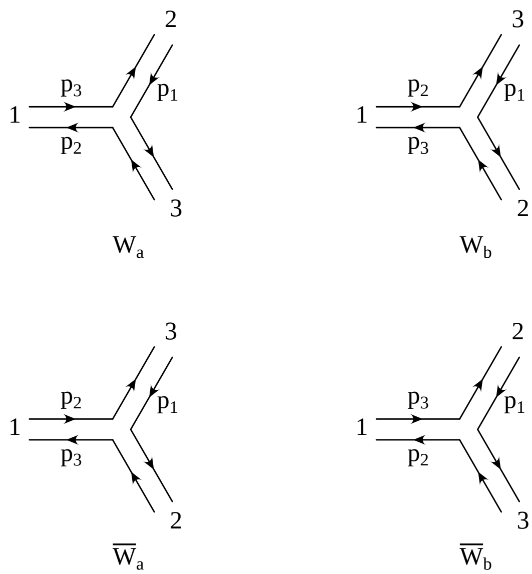

Fig. 2: The vertices W_a, W_b, $\overline{W}_a$, $\overline{W}_b$.

In momentum space, W_a is associated with a factor $e^{ik_1 \wedge k_2}$ where k_i is the momentum associated with Φ_i and $p \wedge q = \Theta^{\mu\nu} p_\mu q_\nu$. Suppose we associate momenta p_i with the lines as shown in Fig. 2 (flowing in the direction of the arrows), so that for W_a, $k_1 = p_3 - p_2$ etc, and for W_b, $k_1 = p_2 - p_3$ etc. Then the exponential factor for W_a can be rewritten using

$$k_1 \wedge k_2 = p_1 \wedge p_2 + p_2 \wedge p_3 + p_3 \wedge p_1 \tag{15}$$

as

$$e^{i \sum_{\text{legs}} p_{\text{out}} \wedge p_{\text{in}}} = \prod_{\text{legs}} e^{i p_{\text{out}} \wedge p_{\text{in}}} \tag{16}$$

where p_{out}, p_{in} are the momenta associated with the lines with arrows pointing out from, or into, the vertex respectively for each leg. We thereby associate an exponential factor with each leg of the vertex. It is easy to check that the exponential factor can also be written in the form Eq. (16) for W_b and indeed for $\overline{W}_a = \operatorname{Tr}\left(\overline{\Phi}_1 * \overline{\Phi}_3 * \overline{\Phi}_2\right)$ and $\overline{W}_b = \operatorname{Tr}\left(\overline{\Phi}_1 * \overline{\Phi}_2 * \overline{\Phi}_3\right)$. Moreover the $\overline{\Phi}_i \Phi_i V^n$ vertex is given by the expression $\operatorname{Tr}\left(\frac{1}{n!} \Phi_i [[\ldots[[\overline{\Phi}_i, V]_*, V]_* \ldots V]_*, V]_*\right)$ with n nested commutators. Again, the exponential factor for one of these vertices can be written in the form given in Eq. (16).

We claim that it is only planar graphs constructed using the vertices above which contribute to the renormalisation-group (RG) functions (β-functions and anomalous dimensions) for the theories with W_1 or W_2 in the noncommutative case. Let us start by considering the theory with W_1. Consider for example the one loop contribution to the anomalous dimension of Φ_1 given by contracting W_1 with $\overline{W}_1$. The contractions of W_a with $\overline{W}_a$, or W_b with $\overline{W}_b$, give planar diagrams, as depicted in Figs 3(a,b), while the contractions of W_a with $\overline{W}_b$, or W_b with $\overline{W}_a$, give non-planar graphs, as depicted in Fig. 3(c,d).

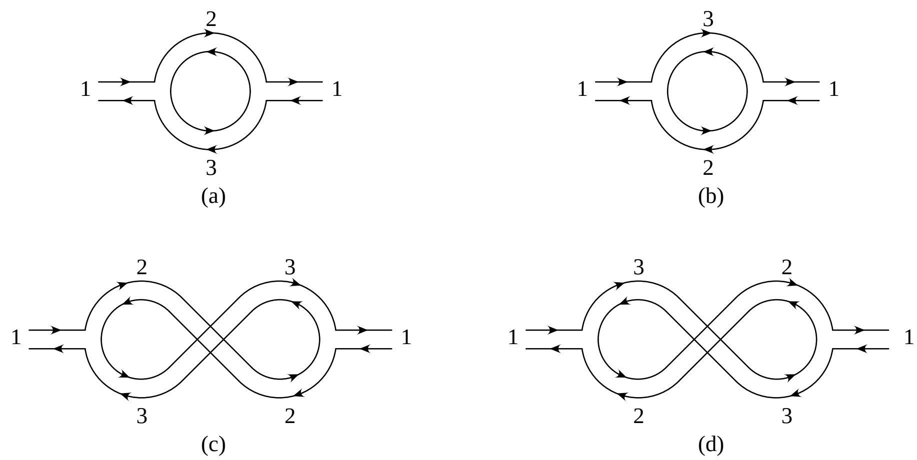

Fig. 3: The one-loop diagrams.

Now these four diagrams all correspond to the same one-loop momentum integral with a single loop momentum. For the planar graphs in Figs. 3(a,b), the loop momentum may be assigned to the closed loop and momenta may be assigned to the other lines consistently with momentum conservation at the two vertices and a given external momentum. It is then clear from Eq. (16) that the exponential factors on the internal pairs of lines cancel in pairs; because the "out" momentum for one vertex is the "in" momentum for its neighbour. The remaining exponential factors from the external legs cancel by momentum conservation. In particular there is no phase factor containing the loop momentum which, if present, would suppress the ultraviolet divergence[1]–[5]. On the other hand, in the case of the non-planar graphs in Figs. 3(c,d), there is no closed loop to which the loop momentum can be assigned, the above argument breaks down, and therefore there will be a phase factor involving the loop momentum (as can easily be checked) making the diagram ultra-violet finite.

This argument readily extends to higher loop orders, to graphs containing gauge fields, and to other RG-functions. For any planar graph, the loop momenta from the corresponding Feynman graph may be assigned to closed loops of the planar graph, and the exponential factors cancel in pairs on internal pairs of lines. In the case of the non-planar graphs, there are fewer planar loops (of the kind apparent in Figs. 3(a,b)) than loop momenta and this argument breaks down. There will then be an overall exponential factor involving at least one of the loop momenta, and this graph will not contribute to the RG-function. Of course a non-planar graph (with a phase factor) may have a planar (and hence divergent) sub-graph, but this graph will be finite after subtraction of sub-divergences; this is analogous to the way that in commutative ϕ^4 theory, the ϕ^6 1PI Green's function is finite, in spite of the fact that it includes 4-point sub-graphs.

We now turn to the theory with superpotential W_2. We shall show that its divergences are the same as those of the theory with superpotential W_1 (with $h_1 \to h_2$). The difference between the superpotentials W_1 and W_2 (apart from $h_1 \to h_2$) lies simply in the sign of W_b (and $\overline{W}_b$). For simplicity we start with diagrams which only contain Yukawa vertices. Note that of course by chirality Ws and $\overline{W}$s must alternate in such a diagram. Once again the only divergent diagrams are the planar ones. Consider any planar diagram. We may assign it an odd or even "parity" according as its sign is changed or unchanged by $W_b \to -W_b$, $\overline{W}_b \to -\overline{W}_b$. We would like to show that every planar diagram has even parity. For simplicity, suppose we join together the external legs of the diagram and imagine it to be drawn on the surface of a sphere. We see that for planar diagrams every closed loop has the same sense of rotation for the arrow (anti-clockwise with our conventions). Therefore the fields Φ_1, Φ_2 and Φ_3 always appear in clockwise order for W_a and anticlockwise order for W_b; and conversely, the fields $\overline{\Phi}_1$, $\overline{\Phi}_2$ and $\overline{\Phi}_3$ always appear in anticlockwise order for $\overline{W}_a$ and clockwise order for $\overline{W}_b$.

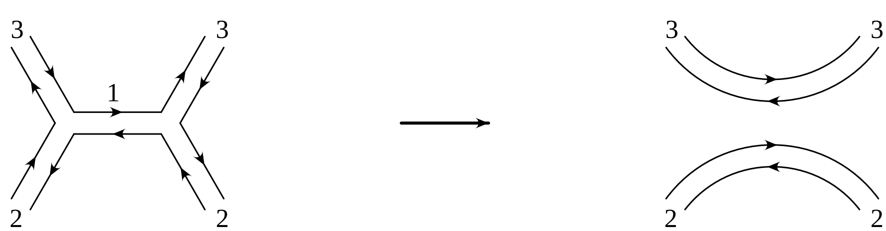

Fig. 4: Reduction of diagrams.

If the diagram contains a pair of linked adjacent vertices W_a and $\overline{W}_a$, or W_b and $\overline{W}_b$, (signalled by a sequence of fields such as $\Phi_2\Phi_1\Phi_2$ in some loop) then we may obtain a graph with two fewer vertices and the same parity by the process depicted in Fig. 4.

We now repeat this process until we can do so no further. The process could terminate in one of two ways: the first possibility is that eventually we obtain a diagram consisting of separate closed loops and no vertices, which clearly has even parity by default; and thus the original diagram must have even parity. The second possibility is that eventually every loop consists of a permutation of the sequence $\Phi_1\Phi_2\Phi_3$ repeated an integral number n times (where n would be even by chirality). But it is easy to see that this is impossible for planar diagrams. The diagram would then consist entirely of hexagons, dodecagons and so on. Suppose we have a diagram with n_6 hexagons, n_{12} dodecagons etc. Let n_V be the number of vertices, n_P the number of propagators and n_L the number of loops. Then we have

$$\begin{aligned} 3n_V =& 6n_6 + 12n_{12} + 18n_{18} + \ldots \Rightarrow n_V = 2n_6 + 4n_{12} + 6n_{18} + \ldots, \\ 2n_P =& 6n_6 + 12n_{12} + 18n_{18} + \ldots \Rightarrow n_V = 3n_6 + 6n_{12} + 9n_{18} + \ldots, \\ n_L =& n_6 + n_{12} + n_{18} + \ldots, \end{aligned} \tag{17}$$

and then

$$n_V - n_P + n_L = -n_{12} - 2n_{18} - \ldots \tag{18}$$

so that Euler's formula

$$n_V - n_P + n_L = 2 - 2\mathcal{G} \tag{19}$$

has no solution for the sphere which has genus $\mathcal{G} = 0$. We deduce that the second possibility does not in fact occur, and therefore the original diagram is indeed of even parity. It is easy to extend this argument to graphs with gauge propagators, by noting that we may remove a gauge propagator without changing the parity of the graph. It follows that the divergences, and thus the RG-functions, of the theory with superpotential W_2 may be obtained from those for superpotential W_1 by replacing h_1 with h_2.

Our main results now follow immediately upon setting $h_1 = h_2 = g$. Firstly, the theory with W_1 now becomes $\mathcal{N} = 4$ NCGT. So we see that the $\mathcal{N} = 4$ NCGT β functions are derived from the planar graphs. We now note that these planar graphs are exactly those which give the leading N contribution to the β-function for the $\mathcal{N} = 4$ CGT, since at L loops they contain the maximum number (L) of closed loops. Since the $\mathcal{N} = 4$ CGT is finite, the leading N contributions must vanish individually at each loop order. Therefore the $\mathcal{N} = 4$ NCGT β functions must also vanish, and $\mathcal{N} = 4$ NCGT is finite to all orders. Secondly, since the RG-functions for the theory with W_2 are identical to those of the theory with W_1, the theory with W_2 is also finite to all orders (for $h_2 = g$).

Clearly the fact that both W_1 and W_2 lead (for $h_1 = h_2 = g$) to finite theories, and the obvious similarity between Eq. (11) and Eq. (12), suggest that W_2 also represents a theory with $\mathcal{N} > 1$ supersymmetry; however we have been unable to demonstrate this. The presence of the commutator in Eq. (6) (as opposed to the anti-commutator in Eq. (7)) is crucial for the additional symmetries (as given, for example, in Ref [25]) associated with the $\mathcal{N} = 4$ invariance. It would clearly be interesting to compare the two theories in the infra-red; it has been argued[7] that the $\mathcal{N} = 4$ theory is free of divergences as $\Theta \to 0$, although such divergences are characteristic of NC theories in general.

By similar reasoning we can use the finiteness of commutative $\mathcal{N} = 2$ theories beyond one loop to establish the corresponding result in the NCGT case, that is for superpotentials of the form

$$W = h \sum_{n=1}^{N_f} \xi_n * \Phi * \chi_n \tag{20}$$

where ξ_n, ϕ, χ_n transform according to the superfield generalisation of Eq. (4), and for $\mathcal{N} = 2$ supersymmetry we require $h = \sqrt{2}g$. The contributions to RG-functions are associated once again with cancellation of phase factors in planar graphs; here these contributions are (at $L \geq 2$ loops) precisely given by the terms of order $N^L, N^{L-1}N_f, N^{L-1}N_f^2, \cdots NN_f^{L-1}$ from the corresponding RG-functions for the CGT. (This corresponds to the Veneziano [26] (as opposed to the 't Hooft) limit, i.e. both $N, N_f \to \infty$ with N/N_f fixed.) Since in the CGT case the RG-functions vanish beyond loop, it follows that all these contributions cancel. By choosing $N_f = 2N$ for one loop finiteness we obtain another class of all orders UV finite theories.

In conclusion: we have established the UV finiteness to all orders of the $\mathcal{N} = 4$ U_N NCGT, a closely related $\mathcal{N} = 1$ theory and the class of one-loop finite $\mathcal{N} = 2$ U_N theories. A simple corollary of our methods is that β_g for the pure non-supersymmetric U_N NCGT is identical to the large-N (or planar) approximation to the β_g for the corresponding SU_N CGT; and for NCQCD (with N_f flavours) the L-loop contribution to the U_N RG-functions are given by the terms from the corresponding commutative SU_N QCD RG-functions of the form $N^a N_f^b$ where $a + b = L$, corresponding once again to the Veneziano[26] limit.

Note Added

After this paper was submitted to the archive we were made aware of some related work:

The UV/IR connection (the existence of infra-red singularities arising from large virtual momenta) was described in Ref. [27]. This paper deals mainly with scalar theories, and in fact describes the cancellation of phase factors involving internal momenta in planar graphs by use of momentum assignments like those shown in our Fig. 2. A rigorous proof of renormalisability for various massive NC scalar theories (in particular $\phi^*\phi\phi^*\phi$ for $d = 4$) was given in Ref. [28]. The relevance of the Veneziano limit for NCQCD described above was remarked in Ref. [29]. A general proof of the renormalisability of a particular supersymmetric noncommutative theory is given for the Wess-Zumino model in Ref. [30]. It was pointed out in Ref. [5] and re-emphasised in Ref. [31] that the divergences of pure U_N noncommutative gauge theory are dictated by the large N limit of the commutative theory. The latter paper also raises the interesting possibility of finite, possibly non-supersymmetric noncommutative theories obtained by orbifold truncation of the $\mathcal{N} = 4$ theory. We also mention the possibility of defining finite noncommutative theories on fuzzy spheres[32]. (See Ref. [33] for the q-deformed case.)

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